An Approximation for Waiting Time Tail Probabilities in Multi-Class Systems

Yuming Jiang, Chen-Khong Tham, Chi-Chung Ko

Abstract—This letter proposes a simple exponential approximation for waiting time tail probabilities in multi-class systems. The approximation is evaluated by making comparisons with simulation results, which show that the proposed approximation is adequate.

Keywords—Waiting time distribution, multi-class system, statistical delay guarantee.

I. INTRODUCTION

Future high speed networks are expected to provide quality of service guarantees, either deterministic or statistical, to multiple classes of applications. A statistical delay guarantee is modeled by $\text{Pr}[\text{Delay} > d] < \varepsilon$, where $d$ is the desired delay bound and $\varepsilon$ is the permissible probability that a packet violates the delay bound. Clearly, to provide such a guarantee, a fundamental issue is to study $\text{Pr}[\text{Delay} > x]$, the tail of the steady-state delay distribution.

To date, the tail of the delay distribution has been extensively studied for single-class systems [1], [2]. However, few results are available for multi-class systems. Thus, analyzing delay distribution in a general multi-class system is extremely important. Suppose $W_i$ is the steady-state waiting time of a class $i$ packet, $\bar{W}_i$ is the average waiting time of class $i$ packets, and $\rho$ is the system traffic intensity. We are suggesting the following approximation:

$$P(W_i > x) \approx \rho e^{-\rho x}/\bar{W}_i,$$

for suitably large $x$.

In the next section, a multi-class system is decomposed into multiple equivalent single-class systems and results for the standard single-class system are utilized to analyze the equivalent systems. Section III derives approximation (1) and Section IV evaluates it. Section V concludes the letter.

II. DECOMPOSITION OF A MULTI-CLASS SYSTEM

Consider a multi-class $GI/GI/1$ queueing system. Suppose that the server is shared by $I$ classes of customers and the waiting room for each class is unlimited. For each class, the interarrival times between customers are independent and identically distributed, taken from a general process, $A_i(x)$, and the service times are independently drawn from a general process, $B_i(x)$. Assume processes $A_i(x)$ and $B_i(x)$ for ($i = 1, ..., I$) are mutually independent. Within each class, customers are served in the First-Come-First-Served manner. Assume that the adopted scheduling discipline is nonpreemptive and work-conserving.

Denote $W_{i,k}$ as the waiting time in queue of the $k^{th}$ class $i$ customer; $\hat{i}$ = all classes ($1, ..., \hat{i} - 1, 1, \hat{i} + 1, ..., I$) other than class $i$; $\hat{k}$ = the number of class $\hat{i}$ customers served before the $k^{th}$ class $i$ customer; $t_{i,k}$ = the interarrival time between the $k^{th}$ and $(k + 1)^{th}$ class $i$ customers; and $s_{i,k}$ = the service time of the $k^{th}$ class $i$ packet. Then, the waiting time of the $(k + 1)^{st}$ class $i$ customer is:

$$W_{i,k+1} = \max\{0, W_{i,k} + s_{i,k} + \sum_{j=k+1}^{\hat{k}+1} s_{\hat{i},j} - t_{i,k}\},$$

where $\sum_{j=k+1}^{\hat{k}+1} s_{\hat{i},j}$ is the delay caused by packets from all other classes served between the $k^{th}$ and $(k + 1)^{st}$ class $i$ packets.

Let us define a new random process as

$$\hat{s}_{i,k} = s_{i,k} + \sum_{j=k+1}^{\hat{k}+1} s_{\hat{i},j}$$

with which, we modify (2) as

$$W_{i,k+1} = \max\{0, W_{i,k} + \hat{s}_{i,k} - t_{i,k}\}.$$  

With (4), an equivalent single-class queueing system for class $i$ customers has been formed, in which the arrival process remains the same as $A_i(x)$, while the service time distribution, denoted by $B_i(x)$, is determined from (3). Since $t_{i,k}$ and $s_{i,k}$ for ($i = 1, ..., I$) and ($k = 1, 2, ...$) are mutually independent, $s_{i,k}$ for ($k = 1, 2, ...$) are also independent of each other based on (3). Hence, the formed equivalent system is a single-class $GI/GI/1$ system and the multi-class system can be decomposed into $I$ number of such single-class systems.

Define $\bar{W}_i = E[W_{i,k}]$, $\bar{t}_i = E[t_{i,k}]$, $\varpi_i = E[s_{i,k}]$, $\hat{s}_{i,k} = E[\hat{s}_{i,k}]$, $\rho_i = \frac{E[A_i(x)]}{E[B_i(x)]}$, and $\rho = \sum_{i=1}^{I} \rho_i$.

In the following, we will utilize results for the standard single-class $GI/GI/1$ system in the formed equivalent system. First, based on the heavy traffic approximation result [3],

$$\bar{W}_i \approx \frac{\sigma_a^2 + \sigma_b^2}{2\rho(1 - \rho)}$$

as $\rho \rightarrow 1$,

where $\sigma_a^2$ and $\sigma_b^2$ are the variance of $A_i(x)$ and $B_i(x)$ respectively. Second, based on the well-known approximation for waiting time tail probabilities in the standard single-class $GI/GI/1$ system [1] [2],

$$P(W_i > x) \approx \alpha e^{-\alpha x},$$

and based on Theorem 4 in [1],

$$\alpha = \eta \bar{W}_i + \alpha(1 - \rho_i^2),$$

as $\rho_i \rightarrow 1$. 

The authors are with the Department of Electrical and Computer Engineering, National University of Singapore, Singapore 119260.
where \( \eta_i \), the asymptotic decay rate, is the unique positive solution to the following transform equation [1]

\[
E^{-sA_i} E^{iB_i} = 1. \tag{8}
\]

Note that, it is difficult to determine \( B_i(x) \) from (3). This implies that it may not be possible to derive \( \eta_i \) from (8). Other methods are needed to find an appropriate value or approximation of \( \eta_i \) in the equivalent single-class system. The following section presents such a method.

III. THE APPROXIMATION

As in [1], assume that as \( s \to 0 \), \( E^{-sA_i} < \infty \) and \( E^{iB_i} < \infty \) for \( i = 1, 2, \ldots, I \). Based on (3), we must also have \( E^{iB_i} < \infty \). Hence, the transforms \( E^{-sA_i} \) and \( E^{iB_i} \) can be expressed in the form of Taylor series expansion. Suppose that as \( s \to 0 \),

\[
E^{-sA_i} = 1 - a_i^{(1)} s + \frac{a_i^{(2)}}{2} s^2 + o(s^3); \tag{9}
\]

\[
E^{iB_i} = 1 + b_i^{(1)} s + \frac{b_i^{(2)}}{2} s^2 + o(s^3). \tag{10}
\]

Here, it is easy to verify that \( a_i^{(k)} \) is the \( k^{th} \) moment of \( A_i(x) \); \( b_i^{(k)} \) is the \( k^{th} \) moment of \( B_i(x) \).

Since \( \eta_i \) is the unique root of (8), using (9) and (10), we express (8) as

\[
\left(1 - a_i^{(1)} \eta + \frac{a_i^{(2)}}{2} \eta^2 + o(\eta^3)\right) \left(1 + b_i^{(1)} \eta + \frac{b_i^{(2)}}{2} \eta^2 + o(\eta^3)\right) = 1
\]

or

\[
1 + \eta_i (\tilde{b}_i^{(1)} - a_i^{(1)}) + \eta_i^2 \left(\frac{a_i^{(2)}}{2} + \frac{b_i^{(2)}}{2} - a_i^{(1)} \tilde{b}_i^{(1)}\right) + o(\eta_i^3) = 1.
\]

Note that by definition, \( a_i^{(1)} = \bar{T}_i, \tilde{b}_i^{(1)} = \bar{s}_i, \rho_i = \frac{\bar{T}_i}{\bar{s}_i}, a_i^{(2)} = \sigma_i^2 + (\bar{T}_i)^2, \) and \( b_i^{(2)} = \sigma_i^2 + (\bar{s}_i)^2 \). After subtracting 1 from both sides and dividing by \( a_i^{(1)} \), the above expression becomes

\[
1 - \tilde{\rho}_i = \eta_i \left(\frac{\sigma_i^2 + \sigma_i^2}{2a_i^{(1)}} + \eta_i \frac{(\bar{T}_i - \tilde{b}_i^{(1)})^2}{2a_i^{(1)}} + o(\eta_i^2)\right) = \eta_i \left(\frac{\tilde{\rho}_i (\sigma_i^2 + \sigma_i^2)}{2\tilde{b}_i^{(1)}} + \eta_i \frac{a_i^{(1)} (1 - \tilde{\rho}_i)^2}{2} + o(\eta_i^2)\right).
\]

Expressing \( \tilde{\rho}_i \) on the right-hand side of (11) in the form of \( 1 - \tilde{\rho}_i \), we obtain

\[
1 - \tilde{\rho}_i = \eta_i \left(\frac{\sigma_i^2 + \sigma_i^2}{2\tilde{b}_i^{(1)}} - \eta_i \frac{\sigma_i^2 + \sigma_i^2}{2\tilde{b}_i^{(1)}} (1 - \tilde{\rho}_i) + \eta_i \frac{a_i^{(1)} (1 - \tilde{\rho}_i)^2}{2} + o(\eta_i^2)\right) = \eta_i \left(\frac{\sigma_i^2 + \sigma_i^2}{2\tilde{b}_i^{(1)}} + \eta_i \frac{a_i^{(1)} (1 - \tilde{\rho}_i)^2}{2} + o(\eta_i^2)\right).
\]

Substituting \( 1 - \tilde{\rho}_i \) in the righthand side of (12), we reduce (12) to

\[
1 - \tilde{\rho}_i = \eta_i \frac{\sigma_i^2 + \sigma_i^2}{2\tilde{b}_i^{(1)}} + o(\eta_i^2). \tag{13}
\]

As in [1], we use the inverse function theorem to determine \( \eta_i \) from (13). Note that in (13), \( 1 - \tilde{\rho}_i \) is in the form of power series of \( \eta_i \). We then express \( \eta_i \) as a power series in \( 1 - \tilde{\rho}_i \), i.e.

\[
\eta_i = \sum_{k=1}^{\infty} c_k (1 - \tilde{\rho}_i)^k. \tag{14}
\]

This leads to coefficient matching in expressions (13) and (14), which yields \( c_1 = \frac{2\tilde{\rho}_i}{\sigma_i^2 + \sigma_i^2} \). Applying this on the first term in the above expression, we obtain (15):

\[
\eta_i = \frac{2\tilde{\rho}_i (1 - \tilde{\rho}_i)}{\sigma_i^2 + \sigma_i^2} (1 + o((1 - \tilde{\rho}_i))) \quad \text{as} \quad \tilde{\rho}_i \to 1. \tag{15}
\]

Combining (15) and (5), the following approximation is obtained:

\[
\eta_i \bar{W}_i \approx \tilde{\rho}_i \quad \text{as} \quad \tilde{\rho}_i \to 1, \tag{16}
\]

from which, the asymptotic decay rate is approximated as

\[
\eta_i \approx \frac{\tilde{\rho}_i}{\bar{W}_i}. \tag{17}
\]

Furthermore, a simple approximation of \( \alpha_i \) follows from (7) and (17):

\[
\alpha_i \approx \tilde{\rho}_i. \tag{18}
\]

Consequently, following from (7), (17) and (18),

\[
P(W_i > x) \approx \tilde{\rho}_i e^{-\tilde{\rho}_i x/\bar{W}_i}. \tag{19}
\]

In the standard single-class system, it is clear that \( \tilde{\rho}_i = \rho \). Also, it is easy to verify that approximation (1) is exact for the single-class \( M/M/1 \) case. In the multi-class system, it is known that the fraction of time that the system is idle is \( 1 - \rho \). Hence, in the equivalent single-class system, the fraction of time that the server is idle must also be equal to \( 1 - \rho \). This implies,

\[
\tilde{\rho}_i = \rho. \tag{20}
\]

with which, approximation (1) is obtained from (19).

In the above derivation, no assumption has been made on the adopted scheduling discipline. This implies that the obtained approximation should hold regardless of the adopted scheduling discipline. Approximation (1) was first observed for a deadline-oriented discipline from simulations [4]. However, no theoretical support was provided in [4]. The next section evaluates approximation (1) under various scheduling disciplines.
IV. EVALUATION

Consider a 4-class system where the service time distribution is deterministic with rate 1. The scheduling disciplines for the evaluation include the Strict Priority (SP), Weighted Fair Queueing (WFQ) and Weighted Round Robin (WRR) disciplines. Figs. 1 and 2 illustrate simulation results as well as approximation results. For Figs. 1 (a) - (e), the interarrival times are exponentially distributed, while for Figs. 1 (f) and 2, each class is loaded with an identical ON/OFF source that has exponentially distributed on and off times. For Fig. 1, $\rho = 0.9$. For Fig. 2, the adopted scheduling discipline is SP and $\rho$ changes from 0.6 to 0.9. For Figs. 1 (a) and (b), $P_i$ in (1) are calculated based on the $M/G/1$ work conservation law [3]; for other sub-figures, simulated values are used.

For Fig. 1 (a), $\rho_1 : \rho_2 : \rho_3 : \rho_4 = 4 : 3 : 2 : 1$; for Fig. 1 (b), $\rho_1 : \rho_2 : \rho_3 : \rho_4 = 1 : 2 : 3 : 4$. For Fig. 1 (c), $\rho_1 = \rho_2 = \rho_3 = \rho_4$, and the assigned weights are 1, 2, 3, and 4; for Fig. 1 (d), $\rho_1 : \rho_2 : \rho_3 : \rho_4 = 1 : 2 : 3 : 4$, and the assigned weights in WRR are 1, 1, 1, and 1. For both Figs. 1 (e) and (f), the two WFQ cases, $\rho_1 = \rho_2 = \rho_3 = \rho_4$, and the weights assigned to each class are 0.1, 0.2, 0.3, and 0.4.

For Fig. 1 (f), the parameters of the ON/OFF source are: on time $T_{on} = 25$, off time $T_{off} = 75$ and peak rate $\Lambda = 0.9$. Similarly, for Fig. 2, $\rho_1 = \rho_2 = \rho_3 = \rho_4$, $T_{on} = 25$, off time $T_{off} = 75$ and peak rate $\Lambda = 0.6, 0.7, 0.8, 0.9$. Due to space limitation, Fig. 2 only presents results for classes 3 and 4.

Figs. 1 and 2 show that for all cases, approximation results are fairly close to the simulation results. The deviation is mostly within 20%. Hence, the approximation is adequate for these scheduling disciplines.

V. CONCLUSION

This letter proposed an approximation for waiting time tail probabilities in multi-class systems, which is a function only of the system traffic intensity and the average waiting time of its corresponding class. In addition, it has been shown that under various scheduling disciplines, the proposed approximation is adequate. However, in the evaluation, the considered tail probabilities are as small as $10^{-2}$. From Figs. 1 and 2, it can be seen that the smaller the tail probability, the larger the deviation. Thus, the proposed approximation in general cannot be called an accurate approximation. To provide a more accurate approximation, further study is needed.

REFERENCES